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Establishment and Application of a Grey Quality Gain–Loss Function Model

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Abstract: Based on Grey System Theory and the inverted normal quality gain–loss function, the inverted normal grey quality gain–loss function model is put forward. According to the constant compensation and hyperbolic tangent compensation, the grey quality gain–loss function model with nominal-type characteristics, larger-the-better characteristics and smaller-the-better characteristics is built. A multivariate grey quality gain–loss function model with multiple sub quality indexes and the concept of grey quality gain–loss cost are proposed. Case analysis is applied to the quality control of dam concrete construction, which verifies the applicability of the model and provides an important reference for research on the new theory of dam concrete construction quality control.

Keywords: dam concrete; construction quality; grey quality gain–loss

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1. Introduction

According to the Japanese expert on quality management, Dr. Genichi Taguchi, as long as the product deviates from the target value, a loss will occur. For the interpretation of this kind of quality loss, he proposed a method to quantitatively describe the product quality loss: the quadratic loss function [1]. Based on the theory of Taguchi's quality loss, scholars conducted various research as follows. The inverted normal distribution function was proposed to use a new quality loss function to solve the unbounded problem existing in Taguchi's model, and an asymmetric quality loss function model was established for the problem of quality loss [2]. Some scholars have put forward a new mathematical method and calculated the proportional constant by studying the quality loss function model [3]. A mfultiple quality loss model was established and the tolerance design method was studied by authors of [4]. A new method of process tolerance design for assembly dimensions with multiple related quality characteristics was raised by other researchers [5]. An optimal process mean model was also established in case the product fails to meet the minimum specification standard [6]. A robust design of the mechanism was developed from the perspective of asymmetric quality characteristics, and a robust design optimization method with asymmetric quality loss has been proposed [7]. Based on Taguchi's quality loss model, the product quality model, including multiple sub quality indexes with relative quality deviation, was studied, and the multivariate form of quality loss model was launched (Fan et al., 2008) [8]. According to the changing demands of the customer, the multivariate quality loss model from Fan Shuhai was improved, and an incremental model of multiple quality loss function was put forward [9]. The relationship between SNR (Signal-to-noise ratio) and robustness of quality loss function model based on engineering practice was studied and verified in [10]. The idea for establishing the double response surface method was adopted and a new multiple response robust optimization design was proposed according to improved quality loss function [11]. A method to optimize the tolerance of products with multiple quality features was released based on the related research of many scholars on multiple quality features [12]. Finally, a quadratic quality loss model with smaller-the-better and larger-the-better characteristics loss was proposed under the premise that the linear term was not neglected [13–16].

As the quality loss function failed to describe the quality compensation effect accurately in production practice, the concept of quality gain–loss function was proposed based on the idea that a constant term in Taylor series expansion means quality compensation. A new dam construction quality loss and gain transfer model based on the GERT (A graphic Review Technology) network was thus constructed and its effective algorithm was designed. The quality gain–loss transfer model and the tolerance optimization method of quality characteristics were studied further [17]. Under the condition that the loss of linear term could not be ignored and the compensation amount remained unchanged, a quality gain–loss function model with larger-the-better and smaller-the-better characteristics was designed [18–21]. Considering the compensation effect in the dam concrete construction based on quality characteristics, a tolerance optimization model of dam concrete construction quality was built, and the adjustment rate of the optimal tolerance of each quality characteristic in each stage of dam concrete construction was studied, so as to maximize the overall quality of construction [22,23]. The process mean design methods of quality gain–loss were discussed under different asymmetric conditions, respectively. The concept of quality gain–loss function (QGLF) is proposed in this paper. A multivariate quality gain–loss function (M-QGLF) was constructed when considering the interaction between multiple quality characteristics. The optimum process mean design method under the condition of quality characteristic following skewed distribution was analyzed based on an example, and the optimum process mean modified formula and deviation degree calculation formula were previously proposed [24]. Furthermore, given that the existing quality gain–loss function failed to describe the boundlessness in engineering practice accurately, a form of inverted normal quality gain–loss function was proposed based on the functional characteristics of the inverted normal function, and larger-the-better and smaller-the-better characteristics and multivariate quality gain–loss function were designed [25]. Also, the fuzziness of quality characteristics was analyzed, the fuzzy quality gain–loss function model was built, and the optimal process mean was designed for asymmetric fuzzy quality gain–loss function [26].

In actual engineering situations, the definition and evaluation of product quality are often based on imprecise intention, and the determination of target value will also be based on approximation, assumptions and engineering experience, which does not have absolute accuracy, which then leads to a certain grey color of quality characteristics, so the quality loss and quality compensation are also imprecise. At the present stage of the proposed research on the quality profit and loss function model, none of them takes into account the greyness of the quality characteristics. Therefore, compared with the known quality gain–loss function model, this paper proposes the idea of grey quality gain–loss and grey quality gain–loss cost on the basis of the inverse normal quality gain–loss function, and researches the calculation method of grey quality gain–loss cost. Under the condition of constant compensation and hyperbolic tangent compensation, the grey quality gain–loss function models featured by larger-the-better and smaller-the-better and multi-element ones are built respectively. Finally, it is applied to the actual dam concrete construction to verify the practicability of the model. Therefore, the establishment of a grey quality profit and loss function model not only enrich the relevant quality theory, but also provide some reference for the actual project quality control.

2. Grey Analysis of Quality Gain Loss Function

2.1. Brief Introduction to Grey System Theory

In cybernetics, the depth of color is often used to describe the clarity degree of information. For example, Ashby referred to objects with unknown internal information as "black box", which is now widely recognized. If the information is unknown, it is represented by "black"; if the information is completely clear, it is represented by "white"; and if some parts of information are clear while others are not, it is represented by "grey". The system with part of its information clear and part of its information unclear is called a grey system. In the process of system research, there are often disturbances inside and outside the system. Coupled with the limited cognitive ability, the information obtained is often uncertain, so it is difficult to fully recognize all the information reflecting the behavior of the system. Only the value range of system elements or parameters can be identified. Usually, the number with the value range instead of the exact value is called grey number. The value range of the grey number can be an interval or a general number set. For instance, an investment project should have a maximum investment limit, and an electrical equipment needs to have a maximum critical value to withstand voltage or current. When the project investment and the allowable values of voltage and current of electrical equipment are all greater than zero, they are called interval grey numbers. If the age scope of a person is from 30 to 35 years old, the person's age may be 30, 31, 32, 33, 34 or 35. Therefore, age is a discrete grey number, and a person's height and weight are continuous grey numbers. Because of the continuity of the quality gain–loss function, grey numbers represent continuous grey numbers in this study.

In social, economic, engineering or scientific research activities, information and data are often incomplete or inaccurate. For example, in agricultural production, it is difficult to accurately estimate the output, output value or loss because of the lack of sufficient information about the natural environment, climate conditions, market demand and labor skills. When adjusting the price system, due to the insufficient forecast of people's psychological endurance, it is impossible to measure the mutual influence caused by the changes in commodity prices, so that it is impossible to adjust the price system properly. For the general social and economic system, given the system itself and its environment, as well as the unclear external and internal boundaries of the system and unclear "internal" and "external" relationship, it is difficult to analyze the relationship between input and output. In addition, uncertainty is also reflected in people's expression of certain objects, ideas or states. For example, user satisfaction determines the quality of the product. The higher the user satisfaction, the better the quality. However, as an experience state, user satisfaction is an inaccurate concept without clear standards, and it is difficult to presently express it accurately with data.

2.2. Grey Analysis of Quality Gain–Loss Function

In the actual production, the definition and evaluation of product quality is often inaccurate, and the determination of target value is often based on experience or approximation, hypothesis rather than accuracy, so there will be grey and even the quality loss and inaccurate quality compensation. For example, it is imprecise that the stress-bearing capacity of the parts in a certain production is not less than 60 MPa, and the rejection rate is not more than 0.02%. In the case of incomplete information and inaccurate data, it is inappropriate to carry out fine modeling. Especially in the actual construction process, if certain process operation standards are established based on certain construction experience, they may fail to accurately describe the real situation. Therefore, the quality gain–loss value calculated according to this grey standard is not accurate, it is grey. In addition, the research of traditional quality gain–loss function on the compensation term is not detailed enough, without considering the universality of the compensation function. In engineering practice, there should be a variety of forms of compensation, including constant compensation and other forms of compensation. In order to be general, we should also consider the variable case of quality compensation and improve the compensation function according to the practical application. In many cases, the quality gain–loss is asymmetric, resulting in the asymmetric quality gain–loss function. Therefore, given the above disadvantages and based on the inverted normal quality gain–loss function, the grey quality gain–loss function model is established, and the grey quality gain–loss function models with larger-the-better, smaller-the-better and multiple characteristics are set up. According to the quality characteristics of normal distribution, the grey quality loss cost is calculated.

3. Grey Quality Gain–Loss Function

3.1. Grey Quality Gain–Loss Function with Nominal-Type Characteristics

Assuming that the target value of a quality characteristic is grey, it can be expressed as grey number $\otimes \in [T_1, T_2]$, where $[T_1, T_2]$ refers to the interval range of grey number, among which the quality loss can be considered as 0, and the grey quality gain–loss function is defined as:

$$\tilde{L}(y) = \begin{cases} \tilde{g}(y) + A\left[1 - exp\left(-\frac{(y-T_1)^2}{2\sigma^2}\right)\right], & y \in [-\infty, T_1) \\ \tilde{g}(y), & y \in [T_1, T_2] \\ \tilde{g}(y) + A\left[1 - exp\left(-\frac{(y-T_2)^2}{2\sigma^2}\right)\right], & y \in (T_2, +\infty] \end{cases} \tag{1}$$

The grey quality gain–loss function model is modified by the piecewise function theory:

$$\tilde{L}(y) = \begin{cases} \tilde{g}(y) + A_1\left[1 - exp\left(-\frac{(y-T_1)^2}{2\sigma_1^2}\right)\right], & y \in [-\infty, T_1) \\ \tilde{g}(y), & y \in [T_1, T_2] \\ \tilde{g}(y) + A_2\left[1 - exp\left(-\frac{(y-T_2)^2}{2\sigma_2^2}\right)\right], & y \in (T_2, +\infty] \end{cases} \tag{2}$$

Among them, T_1 and T_2 are the upper and lower bounds of the interval grey numbers, A_1 and A_2 are the maximum possible loss caused by deviation from the target value on both sides, and σ_1^2 and σ_2^2 are the shape parameters of the quality gain–loss function adjusted on both sides.

(1) The Case of Constant Compensation

The quality compensation keeps constant, that is $\tilde{g}(y) = a$ (constant), and the grey quality gain–loss function of the quality characteristic y is as follows:

$$\tilde{L}(y) = \begin{cases} a + A_1\left[1 - exp\left(-\frac{(y-T_1)^2}{2\sigma_1^2}\right)\right], & y \in [-\infty, T_1) \\ a, & y \in [T_1, T_2] \\ a + A_2\left[1 - exp\left(-\frac{(y-T_2)^2}{2\sigma_2^2}\right)\right], & y \in (T_2, +\infty] \end{cases} \tag{3}$$

Then when $\tilde{g}(y) = a$ (constant), the grey quality gain–loss function $\tilde{L}(y)$ curve is shown in Figure 1.

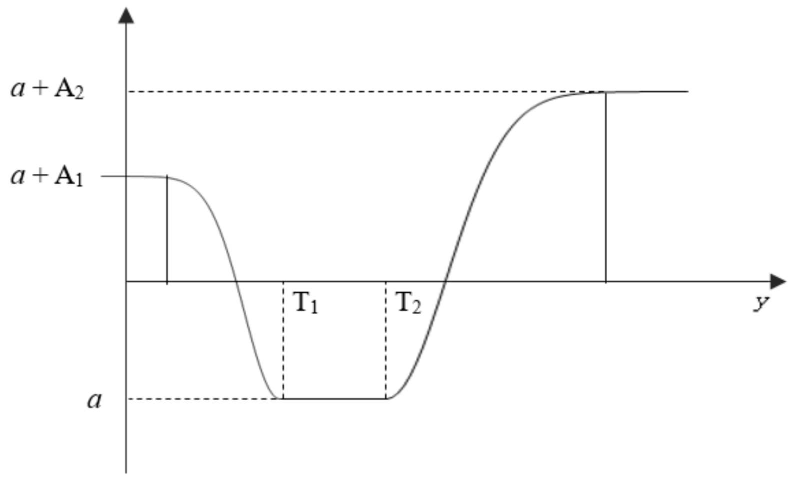

Figure 1. Curve of $\tilde{L}(y)$ with constant compensation.

(2) Hyperbolic Tangent Compensation

According to engineering practice and the characteristics of the hyperbolic tangent function, the compensation function is set as the hyperbolic tangent function. In this case, the grey quality compensation function is as follows:

$$\tilde{g}(y)=\begin{cases}-\alpha\left(1-\dfrac{2}{\exp[2(y-T_1)]+1}\right)+\beta, & y<T_1\\ \beta, & T_1\le y\le T_2\\ \alpha\left(1-\dfrac{2}{\exp[2(y-T_2)]+1}\right)+\beta, & y>T_2\end{cases},\quad \alpha>0 \tag{4}$$

In the case of asymmetric quality compensation, the asymmetric-hyperbolic tangent grey quality compensation function can be expressed as:

$$\tilde{g}(y)=\begin{cases}-\alpha_1\left(1-\dfrac{2}{\exp[2(y-T_1)]+1}\right)+\beta, & y<T_1\\ \beta, & T_1\le y\le T_2\\ \alpha_2\left(1-\dfrac{2}{\exp[2(y-T_2)]+1}\right)+\beta, & y>T_2\end{cases},\quad \alpha>0 \tag{5}$$

The curve of asymmetric-hyperbolic tangent grey quality compensation function is shown in Figure 2.

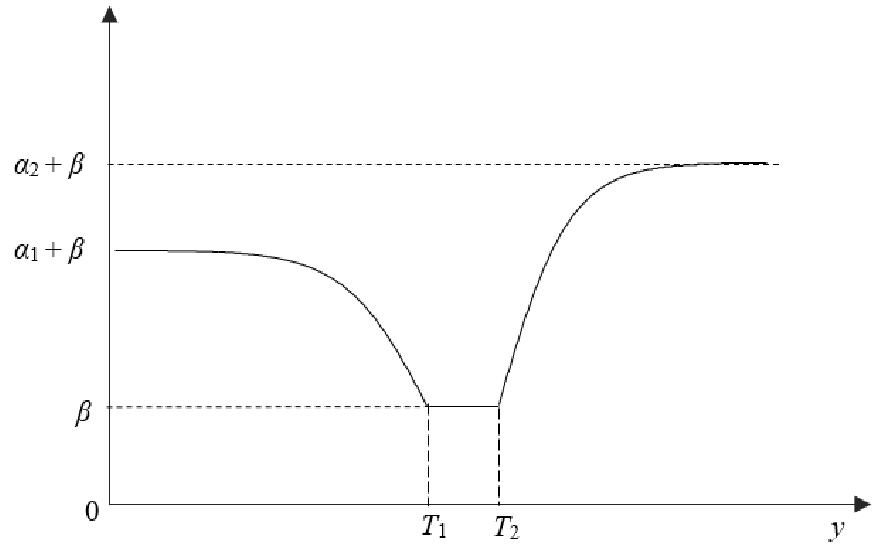

Figure 2. Asymmetric-hyperbolic tangent quality compensation function curve ($\beta > 0$).

The grey quality gain–loss function of asymmetric-hyperbolic tangent compensation can be obtained as follows:

$$\tilde{L}(y)=\begin{cases}-\alpha_1\left(1-\dfrac{2}{\exp[2(y-T_1)]+1}\right)+\beta+A_1\left[1-exp\left(-\dfrac{(y-T_1)^2}{2\sigma_1^2}\right)\right], & y<T_1\\ \beta, & T_1\le y\le T_2\\ \alpha_2\left(1-\dfrac{2}{\exp[2(y-T_2)]+1}\right)+\beta+A_2\left[1-exp\left(-\dfrac{(y-T_2)^2}{2\sigma_2^2}\right)\right], & y>T_2\end{cases} \quad (6)$$

Grey quality gain–loss function of asymmetric-hyperbolic tangent compensation curve is shown in Figure 3.

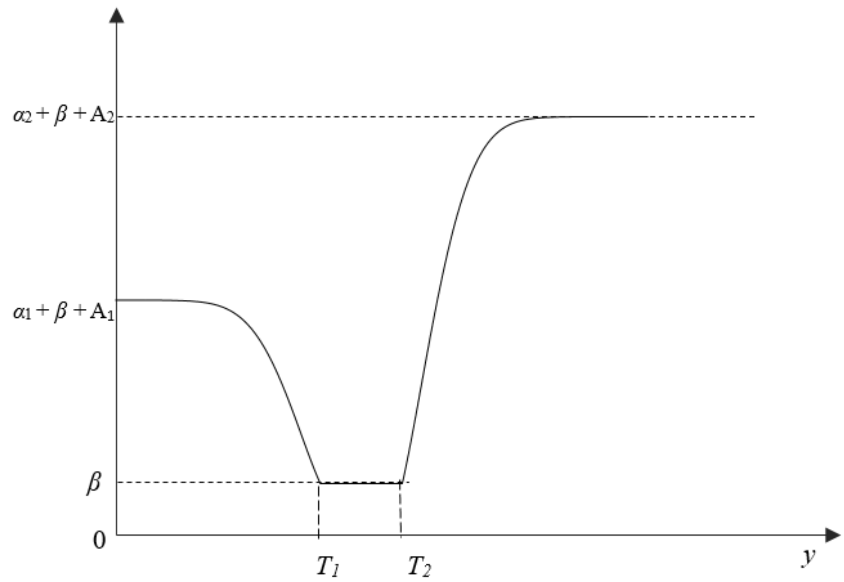

Figure 3. Grey quality gain–loss function curve of asymmetric-hyperbolic tangent compensation ($\beta > 0$).

3.2. Grey Quality Gain–Loss Function with Larger-the-Better Characteristics

The grey number with lower bound and without upper bound is denoted as $\otimes \in [\underline{a}, \infty)$, where $\underline{a}$ means the infimum of grey number $\otimes$, which is a certain number. For example, the quality of a distant celestial body is the grey number with a lower bound, since the quality of the celestial body must be greater than zero, but it is impossible to know the exact value of its quality by general means. If $\otimes$ is used to express the quality of the celestial body, then $\otimes \in [0, \infty)$.

$$\tilde{L}(y) = \begin{cases} \tilde{\rho}(y) + A_l \left\{ 1 - \exp\left[-\frac{(y-\underline{a})^2}{2\sigma_l^2} \right] \right\} & , \quad 0 \le y < \underline{a} \\ \tilde{\rho}(y) & , \quad y \ge \underline{a} \end{cases} \tag{7}$$

Among them, $\underline{a}$ is the infimum of the target value of grey quality characteristics, A_l means the maximum gain–loss caused by deviation from the target value, and σ_l^2 refers to the shape parameter to adjust the loss function.

When the quality compensation keeps constant, i.e., $\tilde{\rho}(y) = a_l$ (constant), the grey quality gain–loss function with larger-the-better characteristics is:

$$\tilde{L}_l(y) = \begin{cases} a_l + A_l \left\{ 1 - \exp\left[-\frac{(y-\underline{a})^2}{2\sigma_l^2} \right] \right\} & , \quad 0 \le y < \underline{a} \\ a_l & , \quad y \ge \underline{a} \end{cases} \tag{8}$$

At this point, the grey quality gain–loss function curve with larger-the-better characteristics is shown in Figure 4.

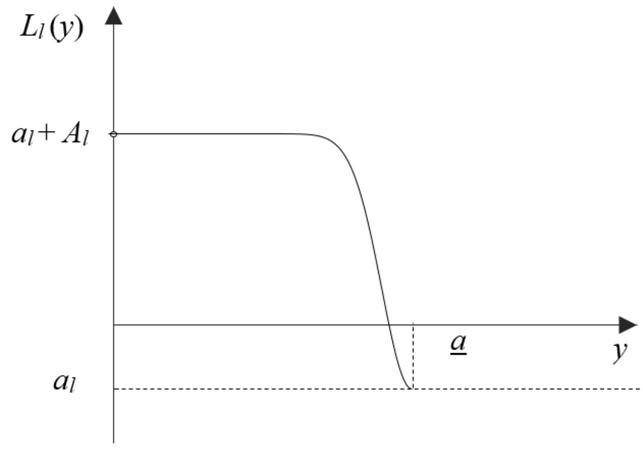

Figure 4. Grey quality gain–loss function curve with larger-the-better characteristics curve ($\tilde{\rho}(y) = a_l$ (constant)).

Similarly, when the compensation is a function of the quality characteristic value y, the hyperbolic tangent compensation function is built based on the properties of the hyperbolic tangent function.

$$\tilde{\rho}(y) = -\alpha \left\{ 1 - \frac{2}{\exp[2(y-\underline{a})] + 1} \right\} + \beta \quad , \quad y \ge 0 \; , \; \alpha > 0 \tag{9}$$

It can be concluded that the grey quality gain–loss function with larger-the-better characteristics of hyperbolic tangent compensation is shown as follows:

$$\tilde{L}(y)=\begin{cases}-\alpha\left\{1-\dfrac{2}{\exp\left[2\left(y-\underline{a}\right)\right]+1}\right\}+\beta+A\left\{1-\exp\left[-\dfrac{\left(y-\underline{a}\right)^2}{2\sigma_l^2}\right]\right\} & , \quad 0\le y<\underline{a}\\ -\alpha\left\{1-\dfrac{2}{\exp\left[2\left(y-\underline{a}\right)\right]+1}\right\}+\beta & , \quad y\ge\underline{a}\end{cases} \tag{10}$$

Figure 5 shows the curve of grey quality gain–loss function with larger-the-better characteristics of hyperbolic tangent compensation.

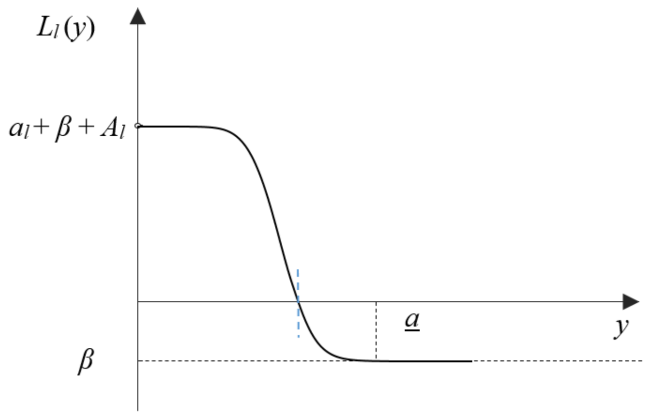

Figure 5. Grey quality gain–loss function curve with larger-the-better characteristics curve (hyperbolic tangent compensation).

3.3. Grey Quality Gain–Loss Function with Smaller-the-Better Characteristics

Some grey numbers with only upper bound but no lower bound can be expressed as $\otimes\in\left[-\infty,\bar{a}\right)$, where $\bar{a}$ means a certain number and also refers to the supremum of grey number $\otimes$. For example, the opposite number of the mass of the celestial body is a grey number with only an upper bound. If $\otimes$ is used to represent the opposite number of the mass of the celestial body, then $\otimes\in\left[-\infty,0\right)$.

In addition, based on smaller-the-better characteristics, when the quality characteristics value reaches 0, the quality gain–loss is minimum and the target value of grey smaller-the-better characteristics can be expressed as grey number $\otimes\in\left[0,\bar{a}\right)$. For example, in the installation process of ordinary concrete reinforcement, the bending of cold extrusion joint of ribbed reinforcement is required to be less than or equal to 4°. The grey number can be read as $\otimes\in\left[0,4\right)$. The grey quality gain–loss function of smaller-the-better characteristics is expected to be:

$$\tilde{L}_s(y)=\begin{cases}\tilde{g}_s(y)+A_s\left\{1-exp\left[-\dfrac{\left(y-\bar{a}\right)^2}{2\sigma_s^2}\right]\right\} & , \quad y>\bar{a}\\ \tilde{g}_s(y) & , \quad 0\le y\le\bar{a}\end{cases} \tag{11}$$

Among them, $\bar{a}$ means the infimum of the target value of grey quality characteristics, A_s is the maximum gain–loss that may be caused by deviation from the target value, and σ_s^2 refers to the shape parameter for adjusting the loss function.

When the quality compensation is constant, i.e., $\tilde{g}_s(y)=a_s$ (constant), the grey quality gain–loss function with smaller-the-better characteristics is expected to be:

$$\tilde{L}_s(y)=\begin{cases}a_s+A_s\left\{1-exp\left[-\dfrac{\left(y-\bar{a}\right)^2}{2\sigma_s^2}\right]\right\} & , \quad y>\bar{a}\\ a_s & , \quad 0\le y\le\bar{a}\end{cases} \tag{12}$$

In this case, the curve of grey quality gain–loss function with smaller-the-better characteristics is shown in Figure 6.

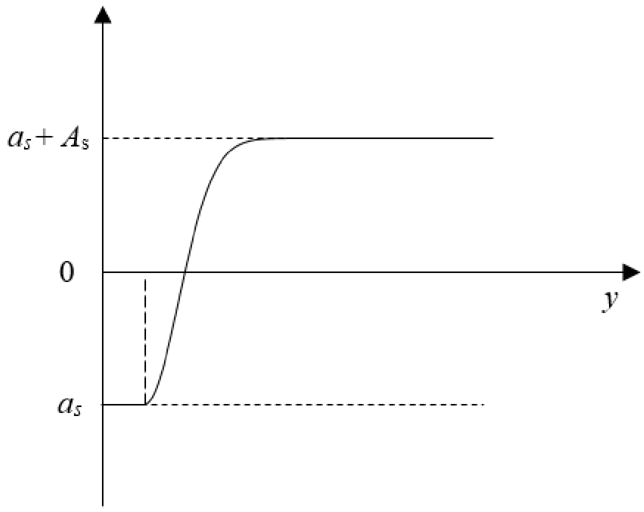

Figure 6. Curve of grey quality gain–loss function curve with smaller-the-better characteristics ($\tilde{g}_s(y) = a_s$ (constant)).

Similarly, when the compensation is a function of the quality characteristic value *y*, the hyperbolic tangent compensation function is built according to the properties of the hyperbolic tangent function.

$$\tilde{g}_s(y) = \alpha\left\{1 - \frac{2}{\exp(2y) + 1}\right\} + \beta \quad , \quad y \geq 0 \quad , \quad \alpha > 0 \tag{13}$$

Therefore, the grey quality gain–loss function with smaller-the-better characteristics of hyperbolic tangent compensation is calculated as follows:

$$\tilde{L}(y) = \begin{cases} \alpha\left\{1 - \frac{2}{\exp\left[2(y - \bar{a})\right] + 1}\right\} + \beta + A\left\{1 - \exp\left[-\frac{(y - \bar{a})^2}{2\sigma_l^2}\right]\right\} & , \quad y > \bar{a} \\ \beta & , \quad 0 \leq y \leq \bar{a} \end{cases} \tag{14}$$

The curve of grey quality gain–loss function with smaller-the-better characteristics of hyperbolic tangent compensation is shown in Figure 7.

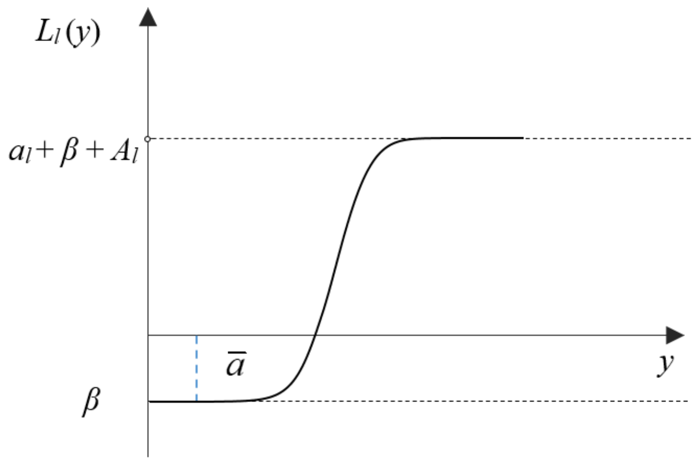

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Figure 7. Curve of grey quality gain–loss function with smaller-the-better characteristics (hyperbolic tangent compensation).

3.4. Multivariate Grey Quality Gain–Loss Function

Assuming several products have grey quality characteristics, including y_1, y_2, $y_3, \ldots, y_I$ respectively, and the target value of quality characteristics y_i is $\otimes_i \in [T_{i1}, T_{i2}]$ (I = 1, 2,…,I), then the corresponding PDF(Probability density function) of quality features are written as $f(y_1)$, $f(y_2)$, $f(y_3)$,..., $f(y_I)$ and the joint PDF of any two quality properties is recorded as $f(y_i, y_j)$ ($i < j$). Grey quality compensation is recorded as $\tilde{g}(y_1, y_2, y_3, \ldots, y_I)$. Multivariate grey quality gain–loss function can be expressed as follows:

$$
\begin{aligned}
\tilde{Q} &= \tilde{g}(y_1, y_2, y_3, \ldots, y_I) + \sum_{i=1, j=1}^{I} A_{ij}\left[1 - f(y_i, y_j)/m_{ij}\right] \\
&= \sum_{i=1}^{I} \tilde{g}(y_i) + \sum_{i=1, j<1}^{I} \tilde{g}(y_i, y_j) + \sum_{i=1}^{I} A_i[1 - f(y_i)/m_i] + \sum_{i=1, i<j}^{I} A_{ij}[1 - f(y_i, y_j)/m_{ij}] \\
&= \sum_{i=1}^{I} \tilde{g}(y_i) + A_i[1 - f(y_i)/m_i] + \sum_{i=1, i<j}^{I} \tilde{g}(y_i, y_j) + A_{ij}[1 - f(y_i, y_j)/m_{ij}] \\
&= \sum_{i=1}^{I} \tilde{L}_i(y) + \sum_{i=1, j<1}^{I} \tilde{L}_{ij}(y)
\end{aligned}
\tag{15}
$$

where $\tilde{g}(y_i)$ is the quality compensation function of quality characteristic y_i, and $g(y_i, y_j)$ is the quality compensation function of mutual effect term; A_i is the maximum quality loss of self-effect term; $f(y_i)$ means the PDF of the quality loss of the self-effect term and the maximum value is m_i; A_{ij} ($i < j$) refers to the maximal quality loss of the mutual effect term; $f(y_i, y_j)$ is the PDF of the quality loss of the mutual effect term and the maximum value is m_{ij}. $\tilde{L}_i(y)$ is the grey quality gain–loss function of self-effect of quality characteristic y_j, and $\tilde{L}_{ij}(y)$ is the grey quality gain–loss function of mutual influence of quality characteristic y_i and y_j. If the quality characteristics y_1, y_2, $y_3, \ldots, y_I$ obey normal distribution, then

$$
\tilde{L}_i(y_i) = \begin{cases}
\tilde{g}(y_i) + A_{i1}\left[1 - exp\left(-\dfrac{(y_i - T_{i1})^2}{2\sigma_{i1}^2}\right)\right], & y_i < T_{i1} \\
\tilde{g}(y_i), & T_{i1} \le y_i \le T_{i2} \\
\tilde{g}(y_i) + A_{i2}\left[1 - exp\left(-\dfrac{(y_i - T_{i2})^2}{2\sigma_{i2}^2}\right)\right], & y_i > T_{i2}
\end{cases}
\tag{16}
$$

$$\tilde{L}_{ij}(y_i,y_j)=\begin{cases}\tilde{g}_{ij}(y_i,y_j)+A_{ij}\left\{1-\frac{1}{\sqrt{2\pi}\sigma_i\sigma_j\sqrt{1-\rho^2}}\,exp\left[\frac{-1}{2m_{ij}(1-\rho^2)}\left(\frac{(y_i-T_{i1})^2}{\sigma_i^2}-2\rho\frac{(y_i-T_{i1})(y_j-T_{j1})}{\sigma_i\sigma_j}+\frac{(y_j-T_{j1})^2}{\sigma_j^2}\right)\right]\right\}, & y_i<T_{i1},y_j<T_{j1}\\ \tilde{g}_{ij}(y_i,y_j), & T_{i1}\le y_i\le T_{i2},\ T_{j1}\le y_j\le T_{j2}\\ \tilde{g}_{ij}(y_i,y_j)+A_{ij}\left\{1-\frac{1}{\sqrt{2\pi}\sigma_i\sigma_j\sqrt{1-\rho^2}}\,exp\left[\frac{-1}{2m_{ij}(1-\rho^2)}\left(\frac{(y_i-T_{i2})^2}{\sigma_i^2}-2\rho\frac{(y_i-T_{i2})(y_j-T_{j2})}{\sigma_i\sigma_j}+\frac{(y_j-T_{j2})^2}{\sigma_j^2}\right)\right]\right\}, & y_i>T_{i2},y_j>T_{j2}\end{cases}\quad (17)$$

3.5. Grey Quality Gain–Loss Cost

When the PDF of product quality characteristic y is known, the average grey quality gain–loss is:

$$E\left[\tilde{L}(y)\right]=\int_{-\infty}^{\infty}\tilde{L}(y)f(y)dy \quad (18)$$

Namely:

$$\begin{aligned}L=&\int_{-\infty}^{T_1}\left\{\tilde{g}(y)+A_1\left[1-exp\left(-\frac{(y-T_1)^2}{2\sigma_1^2}\right)\right]\right\}f(y)dy+\int_{T_1}^{T_2}\tilde{g}(y)f(y)dy\\&+\int_{T_2}^{\infty}\left\{\tilde{g}(y)+A_2\left[1-exp\left(-\frac{(y-T_2)^2}{2\sigma_2^2}\right)\right]\right\}f(y)dy\end{aligned}\quad (19)$$

When the quality characteristic obeys normal distribution (i.e., $Y: N(\mu,\sigma^2)$), the PDF of Y is:

$$f(y)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right],\quad -\infty<y<+\infty \quad (20)$$

The expected gain–loss is:

$$\begin{aligned} E\left[\tilde{L}(y)\right] &= \int_{-\infty}^{T_1}\left\{\tilde{g}(y) + A_1\left[1-\exp\left(-\frac{(y-T_1)^2}{2\sigma_1^2}\right)\right]\right\}\times f(y)dy + \int_{T_1}^{T_2}\beta f(y)dy \\ &+\int_{T_2}^{\infty}\left\{\tilde{g}(y) + A_2\left[1-\exp\left(-\frac{(y-T_2)^2}{2\sigma_2^2}\right)\right]\right\}\times f(y)dy \\ &= \int_{-\infty}^{\infty}\tilde{g}(y)dy + A_1\left\{\begin{array}{l}\Phi\left(\frac{p_2-\mu}{\sigma}\right)- \\ \frac{\sigma_1}{\sqrt{\sigma^2+\sigma_1^2}}\times\exp\left[-\frac{(\mu-p_2)^2}{2(\sigma^2+\sigma_1^2)}\right]\times\Phi\left(\frac{\sigma_1(p_2-\mu)}{\sigma\sqrt{\sigma^2+\sigma_1^2}}\right)\end{array}\right\} \\ &+A_2\left\{\begin{array}{l}1-\Phi\left(\frac{p_3-\mu}{\sigma}\right) \\ -\frac{\sigma_2}{\sqrt{\sigma^2+\sigma_2^2}}\times\exp\left[-\frac{(\mu-p_3)^2}{2(\sigma^2+\sigma_2^2)}\right]\times\left[1-\Phi\left(\frac{\sigma_2(p_3-\mu)}{\sigma\sqrt{\sigma^2+\sigma_2^2}}\right)\right]\end{array}\right\} \end{aligned} \quad (21)$$

When the quality compensation keeps constant, the expected gain–loss is:

$$\begin{aligned} E\left[\tilde{L}(y)\right] &= a + A_1\left\{\Phi\left(\frac{T_1-\mu}{\sigma}\right)-\frac{\sigma_1}{\sqrt{\sigma^2+\sigma_1^2}}\times\exp\left[-\frac{(\mu-T_1)^2}{2(\sigma^2+\sigma_1^2)}\right]\times\Phi\left(\frac{\sigma_1(T_1-\mu)}{\sigma\sqrt{\sigma^2+\sigma_1^2}}\right)\right\}+A_2 \\ &\left\{1-\Phi\left(\frac{T_2-\mu}{\sigma}\right)-\frac{\sigma_2}{\sqrt{\sigma^2+\sigma_2^2}}\times\exp\left[-\frac{(\mu-T_2)^2}{2(\sigma^2+\sigma_2^2)}\right]\times\left[1-\Phi\left(\frac{\sigma_2(T_2-\mu)}{\sigma\sqrt{\sigma^2+\sigma_2^2}}\right)\right]\right\} \end{aligned} \quad (22)$$

When the compensation term is hyperbolic tangent compensation, the expected value of the hyperbolic tangent compensation term cannot be obtained directly by integral operation. Therefore, MATLAB software can be used for calculating the expected compensation part in this case.

As with quality gain–loss, the greater the expected gain–loss, the lower the quality level. Therefore, the average grey quality gain–loss can be regarded as one of the standards for the determination of product quality level.

4. Example Calculation

4.1. Example 1

This example is based on the phase II concrete construction of Danjiangkou Dam heightening project. According to *Evaluation Form and Formfilling Instructions for Unit Works of Water Conservancy and Hydropower Projects* edited by the Construction and Management Department of the Ministry of Water Resources, the research was carried out in combination with the design specifications and design requirements. In the construction quality acceptance and evaluation of ordinary concrete appearance quality inspection process, the cumulative area of pockmarked surfaces and honeycomb per 1000 m^2 should not exceed 5 m^2 for general engineering projects. If treatment measures are implemented in the later stage, the cumulative area can be reduced, and a compensation effect can be produced. According to the engineering practice, the quality characteristic index of the cumulative area of pockmarked surfaces and honeycomb has the grey smaller-the-better quality characteristic, and obeys the normal distribution $N(\mu, 0.5^2)$, the target value can be expressed as $\otimes\in[0,5]$, the maximum loss caused by deviation from the target value reaches 300, the quality compensation is the hyperbolic tangent compensation, the compensation coefficient is $\alpha=1$, the compensation is positive compensation, and the maximum compensation is - 230.

Substituting $\alpha=1$, $\beta=230$, $\bar{a}=5$, $\sigma=0.5$ into Formula (14), we can obtain:

$$\tilde{L}(y)=\begin{cases}1-\dfrac{2}{\exp\left[2(y-5)\right]+1}-230+300\left\{1-\exp\left[-\dfrac{(y-5)^2}{2\times0.5^2}\right]\right\}, & y>5 \\ -230, & 0\le y\le 5\end{cases}$$

It can be seen from Table 1 and Figure 8 that within the target value range of $[0,5]$, there will be no quality loss, and quality compensation will be formed after implementing compensation measures. Within $[0,5]$, the maximum quality compensation is −230, and the quality gain–loss value is −230. When the target value range exceeds $[0,5]$, the quality loss begins to occur, and the quality compensation is carried out in the form of hyperbolic tangent compensation. With the increase of the deviation between y value and the target value, the grey quality gain–loss also increases, and tends toward the maximum value of 71 when $y = 9.5$.

Table 1. Changes of quality gain–loss function $\tilde{L}(y)$.

y	**0**	**5**	**5.5**	**6**	**6.5**	**7**
$\tilde{L}(y)$	−230.000	−230.000	−111.497	30.161	67.572	70.863
y	**7.5**	**8**	**8.5**	**9**	**9.5**	**10**
$\tilde{L}(y)$	70.985	70.995	70.998	70.999	71.000	71.000

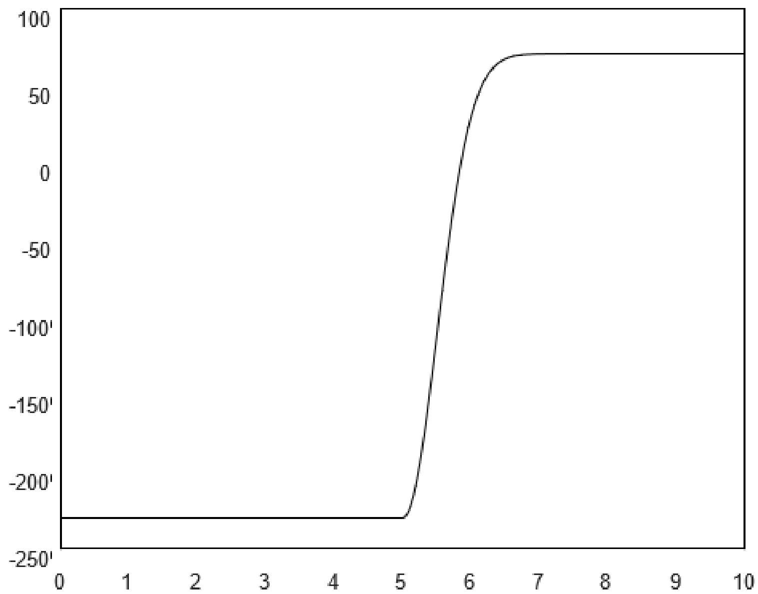

Figure 8. Curve of quality gain – loss function $\tilde{L}(y)$.

4.2. Example 2

This part is also based on the Danjiangkou Dam project as an example. In the foundation treatment and engineering, the dry density of a mud wall is set to be 1.4~1.6/cm³ in terms of the construction quality acceptance evaluation on a single hole splitting grouting process. According to the engineering practice and design requirements, the quality characteristics obey nominal-type and normal distribution, that is, $Y \sim N(1.5, 0.5^2)$. The target value is grey and can be expressed as $\otimes \in [1.4, 1.6]$, and the specification limit is $(LSL, USL) = (0.5, 2.2)$. Correspondingly, the maximum loss caused is $A_1 = 100$ yuan, $A_2 = 70$ yuan, and the quality compensation is fixed and positive, that is, $g(y) = -30$.

It is known that the quality gain–loss reaches the maximum at the specification limit, so the adjustment coefficient is $\sigma_1 = \frac{T_1 - LSL}{4} = \frac{1.4 - 0.5}{4} = 0.225$, $\sigma_2 = \frac{USL - T_2}{4} = \frac{2.2 - 1.6}{4} = 0.15$.

After substituting it with $g\ y\ = a = -30$, $A_1 = 100$, $A_2 = 70$, $T_1 = 1.4$, $T_2 = 1.6$, $\mu = 1.5$ and $\sigma = 0.5$ in Formula (22), we can obtain the following grey quality gain–loss cost:

$$
\begin{aligned}
E\left[\tilde{L}\ y\ \right] = 55 + 100 & \left\{ \begin{array}{l} \Phi\left(\dfrac{1.4-1.5}{0.5}\right) - \dfrac{0.3}{\sqrt{0.5^2+0.225^2}} \\ \times \exp\left[-\dfrac{1.5-1.4\ ^2}{2\ 1.5^2+0.225^2}\right] \times \Phi\left(\dfrac{0.3\ 1.4-1.5}{0.5\sqrt{0.5^2+0.225^2}}\right) \end{array} \right\} \\
+70 & \left\{ \begin{array}{l} 1-\Phi\left(\dfrac{1.6-1.5}{0.5}\right) - \dfrac{0.2}{\sqrt{0.5^2+0.15^2}} \\ \times \exp\left[-\dfrac{1.5-1.6\ ^2}{2\ 1.5^2+0.15^2}\right] \times \left[1-\Phi\left(\dfrac{0.2\ (1.6-1.5)}{0.5\sqrt{0.5^2+0.15^2}}\right)\right] \end{array} \right\} \\
= & 5.639
\end{aligned}
$$

5. Conclusions

The quality evaluation standard is grey in some cases, which leads to the grey calculation of quality profit and loss. Similarly, some quality indicators in the actual construction and production process are also defined as grey in the aspect of quality control of dam concrete construction. Based on the inverted normal quality gain–loss function model, this paper has applied Grey System Theory, put forward the ideas of grey quality gain–loss and grey quality gain–loss cost, and studied the calculation method for grey quality gain–loss cost.

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References

1. Taguchi, G. Quality engineering in Japan. *Commun. Stat.-Theory Methods* **1985**, *14*, 2785–2801.
2. Spring, F.A. The reflected normal loss function. *Can. J. Stat.* **1993**, *21*, 321–330.
3. Khorramshahgol, R.; Djavanshir, G.R. The application of analytic hierarchy process to determine proportionality constant of the Taguchi quality loss function. *IEEE Trans. Eng. Manag.* **2008**, *55*, 340–348.
4. Lee, C.L.; Tang, G.R. Tolerance design for products with correlated characteristics. *Mech. Mach. Theory* **2000**, *35*, 1675–1687.
5. Huang, M.F.; Zhong, Y.R.; Xu, Z.G. Concurrent process tolerance design based on minimum product manufacturing cost and quality loss. *Int. J. Adv. Manuf. Technol.* **2005**, *25*, 714–722.
6. Hunter, W.G.; Karyha, C.P. Determining the most profitable target value for a production process. *J. Qual. Technol.* **1977**, *9*, 176–181.
7. Yang, C.M.; Duan, S.Q. Robust synthesis of function mechanisms by using asymmetric quality loss models. *J. Xihua Univ. (Nat. Sci.)* **2016**, *35*, 30–33.

8. Fan, S.H.; Xiao, T.Y.; Sun, H.; Ni, W.H. The simulation of multivariate product quality model. *Ind. Eng. Manag.* **2008**, *2*, 51–54.
9. Duan, S.S.; Fan, S.H.; Huang, T.H.; Xie, Q.H.; Jiang, N.Y. The research of incremental model based on multivariate quality loss function. *Mach. Des. Manuf.* **2013**, *7*, 253–256.
10. Wang, G.X.; Han, Z.J. Relationship between SN ratio and quality loss of product in case of LTB and STB. *Mech. Sci. Technol. Aerosp. Eng.* **2000**, *19*, 236–238.
11. Wu, J.J.; Xie, Z.W.; Huang, Y.L.; Wu, X.M. A new method for multi-response robust optimization based on improved quality loss function. *Mech. Sci. Technol. Aerosp. Eng.* **2017**, *36*, 519–526.
12. Peng, H.P.; Jiang, X.Q.; Xu, Z.G. Optimal tolerance design based on the quality loss function with multi-correlated characteristics. *China Mech. Eng.* **2008**, *19*, 590–593.
13. Zhang, Y.Y.; Song, M.S.; Han, Z.J. Designing of the quality loss function of the larger the better characteristic under not neglecting the linear term. *J. Appl. Stat. Manag.* **2013**, *32*, 486–491.
14. Zhang, Y.Y.; Song, M.S.; Han, Z.J.; Chen, X.L. Design of quality loss function under the smaller the better characteristic without neglecting the linear term. *Ind. Eng. J.* **2011**, *14*, 81–84.
15. Huang, H.; Huang, M.; Zhang, W.; Yang, S. Experimental study of predamaged columns strengthened by HPFL and BSP under combined load cases. *Struct. Infrastruct. Eng.* **2021**, *17*, 1210–1227.
16. Zhang, W.; Tang, Z. Numerical Modeling of Response of CFRP–Concrete Interfaces Subjected to Fatigue Loading. *J. Compos. Constr.* **2021**, *25*, 04021043.
17. Wang, B.; Li, Z.Y.; Gao, J.Y.; Heap, V. Critical quality source diagnosis for dam concrete construction based on quality gain-loss function. *J. Eng. Sci. Technol. Rev.* **2014**, *7*, 137–151.
18. Wang, B.; Zhou, H.G.; Li, Z.Y.; Nie, X.T. Tolerance optimization for quality characteristics of the dam concrete construction based on quality gain-loss function. *Math. Pract. Theory* **2016**, *46*, 90–99.
19. Huang, H.; Xue, C.; Zhang, W.; Guo, M. Torsion design of CFRP-CFST columns using a data-driven optimization approach. *Eng. Struct.* **2022**, *251*, 113479.
20. Liu, W.; Guo, Z.; Wang, C.; Niu, S. Physico-mechanical and microstructure properties of cemented coal Gangue-Fly ash backfill: Effects of curing temperature. *Constr. Build. Mater.* **2021**, *299*, 124011.
21. Wang, M.; Yang, X.; Wang, W. Establishing a 3D aggregates database from X-ray CT scans of bulk concrete. *Constr. Build. Mater.* **2022**, *315*, 125740.
22. Wang, B.; Fan, T.Y.; Tian, J.; Liu, M.Q.; Nie, X.T. Designing a quality gain-loss function for larger-the-better characteristic and smaller-the-better characteristic under not neglecting the linear term loss and keeping compensation amount constant. *Math. Pract. Theory* **2019**, *49*, 153–160.
23. Liu, C.; Wu, D.; Li, Y.; Du, Y. Large-scale pavement roughness measurements with vehicle crowdsourced data using semi-supervised learning. *Transp. Res. Part C Emerg. Technol.* **2021**, *125*, 103048.
24. Wang, B.; Zhou, H.G.; Li, Z.Y.; Fan, T.Y.; Nie, X.T. Quality gain-loss function and its empirical analysis in dam concrete construction. *J. Coast. Res.* **2020**, *104*, 842–848.
25. Nie, X.T.; Liu, C.; Wang, B. Inverted normal quality gain-loss function and its application in water project construction. *J. Coast. Res.* **2020**, *104*, 415–420.
26. Nie, X.T.; Liu, C.; Guo, W.J.; Liu, M.Q.; Wang, B. Fuzzy quality gain-loss function model and process mean design optimization. *Henan Sci.* **2020**, *38*, 1377–1386.